A Dynamic Programing Analysis of Multiple Guidance Corrections of a Trajectory

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The problem of deciding when to apply guidance corrections to the perturbed trajectory of a spacecraft is treated from the dynamic programing point of view, where the objective of the guidance correction policy is to minimize the expected value of the squared error at the final time, subject to the constraint that the total-correction capability expended be less than some specified value. It is shown that this performance index is related to the probability that the final target error will lie within given limits. With certain constraints on the guidance policy, it is shown that a correction should be performed when a certain switching function passes through zero. Assuming that the orbit-determination procedure has been prespecified, and that the statistics of the correction errors are known, the switching function is found to depend upon the instantaneous state of the system, which is composed of 1) the estimate of the trajectory perturbation to be corrected, 2) the variance of the error in this estimate, and 3) the correction capability of the spacecraft. Equations for computing the switching function are derived, and a numerical example is presented.

Nomenclature†

c = correction capability, m/sec
$E[a^2]$ = variance of proportional type of execution error (dimensionless)
$E[b^2]$ = variance of nonproportional type of execution error (m/sec)^2
$E[\,]$ = statistical expectation (average value) over all similar experiments of the quantity in brackets
k = sigma level of simulated estimate
m_i^* = minimum-variance estimate of target error predicted at t_i, kms
q_1, q_2 = constants in exponental approximation of $g(\lambda)$
t_f = final decision time, sec
t_i = time at ith decision point, sec
v = (const) speed of spacecraft toward target, m/sec
$[x|y]$ = value x given that y occurs
Δv_i = velocity impulse correction perpendicular to direction of motion applied at t_i, m/sec
α_i = variance of error in estimate m_i^*, assuming no correction at t_i, km^2
β_i = variance of error in estimate m_i^*, given a correction at t_i, km^2
γ_i = variance of error in the estimate m_{i+1}^*, considering only orbit determination data in the interval $\{t_i, t_{i+1}\}$, km^2
Δt = time between decision points, sec
η_f = same as η_i, with t_f replacing t_i
η_i = variance of error in estimate m_i^*, considering only orbit determination data in the interval $\{t_0, t_i\}$, km^2
ρ_i = variance of error in estimate m_f^*, considering only orbit determination data in the interval $\{t_i, t_f\}$, km^2
$\sigma\theta$ = standard deviation of uncorrelated noise on each angular observation of (dimensionless) star angle (Fig. 1)
τ_i = time-to-go to closest approach, evaluated at t_i, sec
ψ_i = variance of estimate m_f^*, assuming a correction only at t_i and considering all orbit determination data, km^2
ω_i = variance of error in estimate m_f^*, assuming a correction only at t_i and considering all orbit determination data, km^2
(*) = an estimated quantity

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† See Table 1 for equations describing the quantities defined here. Boldface symbols represent column vectors; a matrix is denoted by a capital letter. Unless otherwise indicated, variance (or uncertainty) refers to the variance of the error in the estimate (as distinguished from the variance of the estimate itself).

I. Introduction

A SPACECRAFT traversing a coast trajectory toward some target region in space is guided to its final destination by applying one or more small velocity impulse corrections (maneuvers) at certain times along the path to null the predicted target error. The prediction (estimate) of the target error is achieved by an orbit-determination process; the required corrections are computed using linear perturbation theory, and the impulse is delivered by a rocket motor, which applies an acceleration to the spacecraft for a relatively short period of time. The selection of times for performing the velocity corrections to the orbit, and the determination of what fraction of the predicted target error is to be nulled by each correction is termed the *guidance policy*. It is the purpose of this paper to develop a guidance policy that will minimize the expected value of the squared target error, subject to the constraint that the total propellant expended in performing the corrections is less than some prespecified amount.

Defining the guidance policy is an easy task if the orbit is perfectly known, if the correction can be made perfectly, and if there is adequate correction capability (propellant). Otherwise the policy is not readily constructed. There are factors that tend to cause a maneuver to be made early, such as the smaller amount of correction capability required to null a given target error; and there are factors that tend to cause it to be made late, such as the need to process more data to get a better estimate of the orbit. The random errors arising in the execution of the correction must be considered, since they affect the uncertainty in the knowledge of the orbit parameters. The problem, then, is to develop a guidance policy that will allot the given correction capability in a way that will cause some performance index to be minimized, taking into account the uncertainty arising from orbit-determination and execution errors.

The theory concerning the single-impulse correction is well known and was implemented in the successful Mariner II fly-by mission to the planet Venus.[1] In this case, a suitable single-manuever time is chosen from preflight studies of orbit-determination and execution error statistics, and the correction capability to be carried aboard the spacecraft is determined by mapping the covariance matrix of injection guidance errors to the selected maneuver point to obtain the covariance matrix of velocity-to-be-gained components. The situation becomes much more complex when more than one maneuver is considered, for then the future guidance and tracking policy must be considered in performing a correction at any given time. It becomes necessary, in general, to consider both the present and future uncertainty in the knowledge of the orbit, and the errors in the measurement devices being used to determine the orbit. The target error criterion and desired accuracy must be defined, as well as the bound on the total velocity correction that can be applied. This important inquiry has recently received considerable attention by treating it as an optimization problem and has been attacked from several different points of view by Battin,[2] Breakwell,[3] Striebel,[4] and Lawden.[5] The analysis presented here approaches the problem from the dynamic programing point of view,[6] defining an optimal policy as one that minimizes the mean-squared target error, subject to constraints on the total correction capability that can be allotted. The guidance policy is adaptive in the sense that at any decision time t_i it is dependent upon the estimate of the error to be corrected, the uncertainty in this estimate, and the correction capability available, all of which are time-varying random variables over the ensemble of all perturbed trajectories.

II. Summary

An idealized guidance problem is defined, assuming that a series of velocity impulse corrections are to be applied to the trajectory of the spacecraft while it is traveling in a straight line toward impact on a massless planet. The equations describing the orbit determination and guidance correction(s) applied are presented. It is assumed that the orbit determination policy is prespecified, i.e., the types of observed data to be gathered throughout the entire mission, and the times for making these observations are known from preflight studies and do not depend upon the guidance policy. The statistics of the errors arising from executing the corrections are assumed known.

The performance index p_i to be minimized at any time t_i is defined as the expected value of the sum of the orbit determination uncertainty immediately after the final correction (at prespecified final time t_f) plus the square of the error uncorrectable because of depleting the correction capability prior to t_f, i.e.,

$$p_i = E[\beta_f + r^2 | \text{ all corrections } t_i \ldots t_f] \quad (1)$$

where β_f is the final orbit-determination variance and r is the estimate of the target error immediately following the correction at t_f. The case $r \neq 0$ occurs when there is insufficient correction capability at t_f, and total correction of the estimated error cannot be made. The motivation for choosing this particular performance index is given in Sec. IV.

A sequence of *decision times* $t_i < t_f$ is defined along the trajectory, where the possibility of performing a correction is to be examined. The state of the system $\mathbf{x}$ at any time t_i is considered to be composed of 1) the minimum variance estimate (prediction) of the uncorrected target error m_i^*, which is obtained from the orbit-determination process by considering all data (including the *a priori* estimate) gathered prior to t_i; 2) the variance of the error in this estimate; and 3) the amount of velocity correction capability that can be allotted to the remainder of the mission. The optimization problem is formulated from the dynamic programing point of view, and "two-correction" and "total-correction" constraints are imposed upon the guidance policy. It is shown that the resulting policy is implemented at time t_i in the following steps:

1) Calculate the performance index corresponding to total corrections only at t_i and t_f, i.e.,

$$[p_i|i,f] = E[\beta_f + r^2| \text{ corrections only at } t_i \text{ and } t_f] \quad (2)$$

2) Calculate the performance index corresponding to a total correction only at t_f, i.e.,

$$[p_i|0,f] = E[\beta_f + r^2| \text{ correction only at } t_f] \quad (3)$$

3) If $[p_i|i,f] - [p_i|0,f] \geq 0$, make no correction at t_i; go on to next decision time t_{i+1}. If the inequality does not hold, go on to step 4.

4) Calculate the performance index corresponding to total corrections only at t_{i+1} and t_f, i.e.,

$$[p_i|i+1,f] = E[\beta_f + r^2| \text{ corrections only at } t_{i+1} \text{ and } t_f] \quad (4)$$

This computation is made possible at t_i by recognizing that the expected value of the estimate of the target error m_{i+1}^* at t_{i+1} is the current estimate, i.e.,

$$E[m_{i+1}^*| \text{ no correction at } t_i] = m_i^*$$

The orbit determination uncertainties at t_{i+1} can be computed.

5) Form the switching function

$$s_i = [p_i|i,f] - [p_i|i+1,f] \quad (5)$$

If s_i is positive, no action is taken. If it is negative or zero, a total correction is applied at t_i.

6) When the next decision time is reached, the process is reinitiated, this time with a new estimate of the error m_{i+1}^*, based upon the action taken at t_i and the tracking data received during the interval.

The case of insufficient correction capability to accomplish the mission and the case of a limited number of corrections are discussed. Numerical results are presented, and the extension to a more general case is discussed.

III. Description of the Idealized Guidance Problem

The essential ideas of this paper are developed by considering the idealized one-dimensional problem described below. In Sec. VIII the extension of the problem to the more general case is discussed.

The one-dimensional problem is constructed by imagining that the spacecraft is moving in a zero-gravity field at known speed v toward a massless target, and the time-to-go to closest approach is known. A series of velocity impulse corrections perpendicular to the direction of motion can be accomplished at any or all of the prespecified decision times ($t_0, t_1, \ldots, t_f$), where t_0 is the start of the problem and t_f is the final time. The objective of the guidance system is to minimize the expected value of the final squared target error (Fig. 1). The correction made at t_i is

$$\Delta v_i = -(d_i m_i^*/\tau_i) \quad (6)$$

where m_i^* is the estimate of the target error at t_i, obtained from the orbit-determination process; τ_i is the time-to-go to closest approach at t_i; thus $\tau_i = (t_{\text{closest approach}} - t_i)$; and d_i is the decision variable that determines the fraction of the estimate to be nulled at t_i ($0 \leq d_i \leq 1$).

Between any two decision times t_i, t_{i+1}, the minimum-variance estimate of the target error Δm_i^* is obtained from the

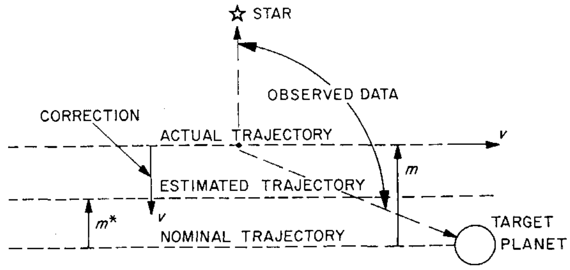

Fig. 1 Idealized guidance problem.

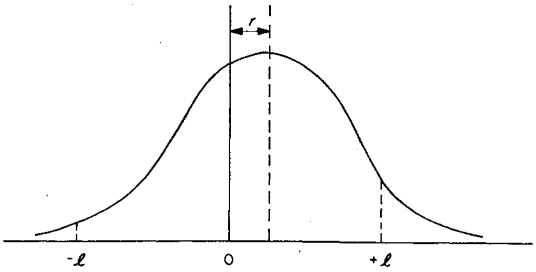

Fig. 2 Biased probability density function.

orbit-determination process in the interval. The variance of the error in that estimate is γ_i. If m_i^* was the previously obtained minimum-variance estimate at t_i, with variance α_i, the combined estimate at t_{i+1} is

$$m_{i+1}^* = [\alpha_i^{-1} + \gamma_i^{-1}]^{-1}[\alpha_i^{-1}m_i^* + \gamma_i^{-1}\Delta m_i^*] \quad (7)$$

The variance of the combined estimate is

$$\alpha_{i+1} = [\alpha_i^{-1} + \gamma_i^{-1}]^{-1} = [\alpha_i\gamma_i/(\alpha_i + \gamma_i)] \quad (8)$$

At time t_0, the m_i^* and α_i are the *a priori* values.

If a correction is made at t_i, there will be further uncertainty added to the knowledge of the target error because of the random execution errors that arise in accomplishing the correction. Thus,

$$\beta_i = \alpha_i + E[a^2](d_i m_i^*)^2 + E[b^2]\tau_i^2 \quad (9)$$

where β_i is the target error variance immediately after the correction at t_i, $E[a^2]$ is the variance of the proportional type of execution error (expressed as a decimal fraction), and $E[b^2]$ is the variance of the nonproportional type of velocity execution error (expressed in m^2/sec^2). The assumption will be made that the execution error causes a transverse position displacement without affecting the uncertainty in the direction of the velocity vector, thereby simplifying the subsequent orbit determination process. If a correction is made at t_i, the quantity β_i is substituted for α_i in Eqs. (7) and (8).

For the purpose of the subsequent analysis, it is necessary to predict the orbit-determination uncertainty at t_f, given that total corrections are made only at t_i and t_f, and considering the orbit-determination data that are to be gathered between t_i and t_f. This quantity is

$$[\omega_i|i,f] = E[\alpha_f|\text{ corrections at } t_i \text{ and } t_f] = [(1/\beta_i) + (1/\rho_i)]^{-1} = [\beta_i\rho_i/(\beta_i + \rho_i)] \quad (10)$$

where ρ_i is the variance of the error in the estimate corresponding to the data gathered between t_i and t_f, given by

$$\rho_i^{-1} = \sum_{j=i}^{t_f - 1} \gamma_j^{-1} \quad (11)$$

The variance of the estimate at t_f (as distinguished from the error in the estimate) predicted at t_i is

$$[\psi_i|i,f] = E[m_f^{*2}|\text{ correction only at } t_i] = \beta_i - \omega_i = [\beta_i^2/(\beta_i + \rho_i)] \quad (12)$$

The quantities ω_i and ψ_i for the case of corrections only at t_f, or only at t_{i+1} and t_f, are obtained in a similar fashion, as described in Sec. VI. The use of these results for determining the policy is developed below.

IV. Performance Index

The performance index defined by Eq. (1) is the expected value of the squared target error at the final time t_f. The motivation for choosing this particular criterion is that the resultant guidance policy effectively maximizes the probability that the final target error will be within some limits $\pm l$ if it is assumed that the final residual error is always small relative to the standard deviation. This conclusion is verified below.

Suppose r and $\nu[\nu = (\beta_f)^{1/2}]$ are, respectively, the mean and standard deviation of the normal distribution of the final target error m_f (Fig. 2). Then

$$\text{prob }(-l \le m_f \le l) = \int_{-[(l+r)/\nu]}^{+[(l-r)/\nu]} f(z)dz \quad (13)$$

where l is a given limit, and

$$f(z) = [1/(2\pi)^{1/2}] \exp -(z^2/2) \quad (14)$$

If r is assumed to be always small relative to ν, Eq. (13) may be written

$$\text{prob }(-l \le m_f \le l) - \int_{-(l/\nu)}^{+(l/\nu)} f(z)dz \cong \int_0^{r/\nu}\left[f\left(\frac{-l}{\nu}\right) + z\frac{df}{dz}\left(\frac{-l}{\nu}\right)\right]dz - \int_0^{r/\nu}\left[f\left(\frac{l}{\nu}\right) + \frac{df}{dz}\left(\frac{l}{\nu}\right)\right]dz = -\left[\left(\frac{l}{\nu}\right)\left(\frac{r}{\nu}\right)^2\right]f\left(\frac{l}{\nu}\right) \quad (15)$$

But if $p = \nu^2 + r^2 = \beta_f + r^2$, where r^2 is small, then by expanding about $r = 0$, it follows that ‡

$$\int_{-l/(p)^{1/2}}^{+l/(p)^{1/2}} f(z)dz - \int_{-(l/\nu)}^{+(l/\nu)} f(z)dz \cong 2f\left(\frac{l}{\nu}\right) \times \left[r\frac{\partial}{\partial r}\left(\frac{l}{p^{1/2}}\right)_{r=0} + \frac{r^2}{2}\frac{\partial^2}{\partial r^2}\left(\frac{l}{p^{1/2}}\right)_{r=0}\right] = -\left[\left(\frac{l}{\nu}\right)\left(\frac{r}{\nu}\right)^2\right]f\left(\frac{l}{\nu}\right) \quad (16)$$

For any given values of l and ν, the left-hand side of Eq. (16) is clearly maximized by minimizing p, which implies that the left-hand side of Eq. (15) is also approximately maximized. This establishes the desired relationship. Since only the expected value of p can be computed at t_i, the performance index given by Eq. (1) is a reasonable one. (It should be noted, however, that assuming r small is not equivalent to assuming $E[r^2]$ small.)

Anticipating the analysis to follow, suppose that a total correction ($d_i = 1$) is made at t_i, and consider the evaluation of the expected value of p_i given that there are no further corrections until t_f.

The correction capability remaining to be applied at the final time t_f then becomes

$$c_f = c(t_f) = c(t_i) - (m_i^*/\tau_i) \quad (17)$$

Any estimate $m_f^* \le c_f\tau_f$ can be nulled at t_f, resulting in $r(m_f^*)$, as shown in Fig. 3. Thus, the expected value of r^2, evaluated at t_i, is

$$E[r^2|\text{ corrections at } t_i \text{ and } t_f] = 2\psi_i \int_{\lambda_i}^{\infty} f(z)(z - \lambda_i)^2 dz \quad (18)$$

where

$$\lambda_i = c_f\tau_f/(\psi_i)^{1/2} \quad (19)$$

and ψ_i is defined by Eq. (12). Thus, the expected value of p_i, evaluated at t_i by assuming a total correction at t_i and t_f, is

$$[p_i|i,f] = \psi_i g(\lambda_i) + \omega_i + \psi_i E[a^2] + \tau_f^2 E[b^2] \quad (20)$$

where the *residual function* is (Fig. 4)

$$g(\lambda) = 2\int_{\lambda}^{\infty} f(z)(z - \lambda)^2 dz \quad (21)$$

and ω_i is defined by Eq. (10). The calculation of p_i for cor-

‡ This equivalence of bias and standard deviation was pointed out in an unpublished paper by T. W. Hamilton of the Jet Propulsion Laboratory.

rections only at t_f, or only at t_{i+1} and t_f, is made in a similar fashion, as will be described in Sec. VI.

V. Dynamic Programing Formulation

The guidance policy, which minimizes the performance index discussed in the previous section, can be formulated by invoking the principle of optimality of dynamic programing,[6] which states that an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

This principle is applied here by imagining a set of tables at each time t_i which presents the minimum value of the performance index p_i and the associated decision variable d_i as a function of the state variables (the predicted target error m^*, the variance of the error in this estimate α, and the correction capability c) of the system. These tables would be constructed by working backward from the final time, at each t_i considering all conceivable combinations of state variables. At each t_i, the decision and index are arrived at by finding the decision that will transfer the state to the subsequent decision time t_{i+1} in such a way as to obtain minimum p_{i+1}, which is evaluated by interpolating the state variables in the previously computed table at t_{i+1}. The mathematical formulation is as follows. Let d_i = the decision at t_i, i.e., the fraction of the estimated miss to be corrected ($0 \leq d \leq 1$); $\mathbf{x}_i$ = the state of the system at time t_i; i.e.,

$$\mathbf{x}_i{}^T = \{m,{}^*\alpha,c,\}_i \tag{22}$$

$\min p_i(\mathbf{x}_i)$ = the value of the performance index resulting from starting in state $\mathbf{x}_i$ at t_i and employing an optimal policy until the final time t_f. If the trajectory is divided into a sequence of decision times ($t_0, t_1, \ldots, t_i, \ldots, t_f$), where the option of making a correction is available, then§

$$\min p_i(\mathbf{x}_i) = \min_{d_i}\{E[\min p_{i+1}(\mathbf{x}_i)]\} \tag{23}$$

Assuming that $\min p_{i+1}(\mathbf{x}_{i+1})$ is approximately a linear function of $m_{i+1}{}^*$, it follows that $E[\min p_{i+1}(\mathbf{x}_i)]$ = the value of the performance index resulting from starting in state $\hat{\mathbf{x}}_{i+1}$ at t_{i+1} and employing an optimal policy until the final time t_f, where

$$\hat{m}_{i+1}{}^* = E[m_{i+1}{}^*] = (1 - d_i)m_i{}^* \tag{24}$$

$$\hat{\alpha}_{i+1} = \begin{cases} [\beta_i\gamma_i/(\beta_i + \gamma_i)] & \text{if} \quad d_i > 0 \\ [\alpha_i\gamma_i/(\alpha_i + \gamma_i)] & \text{if} \quad d_i = 0 \end{cases} \tag{25}$$

$$\hat{c}_{i+1} = c_i - (d_i m_i{}^*/\tau_i) \tag{26}$$

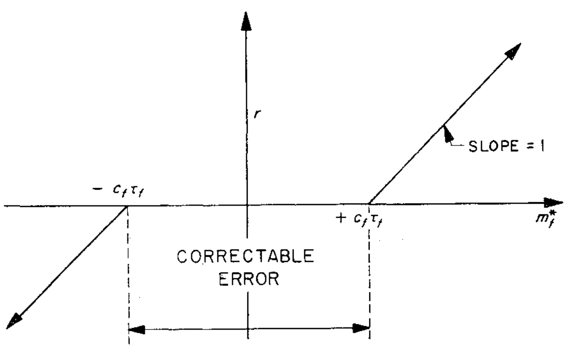

Fig. 3 Residual error at final time.

§ It is assumed that there is sufficient correction capability at t_i to perform a total correction, i.e., $d_i = 1$ is a legitimate case. The case of insufficient correction capability is discussed in Sec. VII.

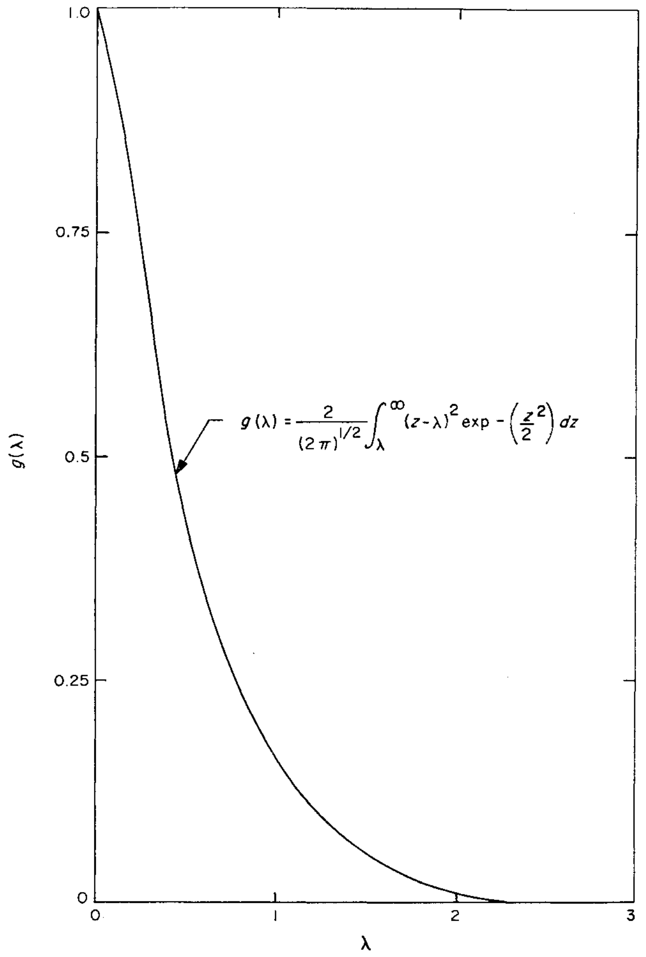

Fig. 4 Residual function.

The α_i, β_i, and γ_i are defined in Sec. III. At the final time t_f,

$$\min p_f(\mathbf{x}_f) = \min_{d_f} \{[\alpha_f + (m_f{}^*)^2], \\ [\beta_f + (1 - d_f)^2(m_f{}^*)^2]\} \tag{27}$$

The process of generating the tabular function, $\min p_i$, and the associated decision variable d_i as a function of the three state variables and the time could present a difficult computational problem, but it is shown in Sec. VI that the guidance policy can be determined quite simply if certain constraints are imposed.

VI. Guidance Policy with Constraints

The optimal guidance policy can be determined relatively simply at each decision time t_i if the following constraints are imposed: 1) all of the estimated (predicted) target error is to be nulled each time a correction is made; that is, no partial corrections are allowed; and 2) every correction decision ($d = 1$) is based upon the assumption that no further corrections will be required until the final time t_f.

Thus, each decision is based upon total- and two-correction policy, where:

Definition 1: A total-correction policy assumes at each decision time t_i that either no correction ($d_i = 0$) or total correction ($d_i = 1$) is to be accomplished.

Definition 2¶: A two-correction policy assumes at each decision time t_i that at most two corrections will be accom-

¶ Note that more than two corrections on any given trajectory may be performed, for the two-correction policy is reapplied at t_{i+1} even if $d_i = 1$. The new orbit determination information obtained after t_i may demand still another correction before t_f.

Table 1 The guidance policy logic

Input: $\tau_0, v, \Delta t, \alpha_0, \sigma_\theta, k, c, E[a^2], E[b^2], \eta_f, \tau_f, q_1, q_2$

Enter at time t_i, where $t_0 < t_i < t_f$

Let $\tau_i = \tau_0 - i\Delta t$

Proceed as follows:

A. Orbit determination computations

$$\gamma_{i-1} = (\sigma_\theta v \tau_{i-1})^2$$
$$\alpha_i = (\alpha_{i-1})(\gamma_{i-1})(\alpha_{i-1} + \gamma_{i-1})^{-1}$$
$$\eta_i^{-1} = \sum_{j=0}^{i-1} \gamma_j^{-1}$$
$$\rho_i = (\eta_f \eta_i)(\eta_i - \eta_f)^{-1}$$

B. Computation of simulated estimate (see Part IX)

$$m_i^* = k(\alpha_0 - \alpha_i)^{1/2}$$

C. Test for propellant depletion (see Part VII)

$$\Delta_i = c\tau_i - m_i^*$$

If $\Delta_i \leq 0$, go to propellant depletion mode of operation (Part VII)

If $\Delta_i \geq 0$, continue

D. Penalty for correction at t_i

$$\beta_i = \alpha_i + (m_i^*)^2 E[a^2] + \tau_i^2 E[b^2]$$
$$\omega_i = (\beta_i \rho_i)(\beta_i + \rho_i)^{-1}$$
$$\psi_i = \beta_i - \omega_i$$
$$\lambda_i = [c - (m_i^*)(\tau_i)^{-1}](\tau_f)(\psi_i)^{-1/2}$$
$$g_i = \exp - (q_1\lambda_i + q_2\lambda_i^2) \text{ (see Eq. 47)}$$
$$[p_i|i,f] = \omega_i + \psi_i g_i + \psi_i E[a^2] + \tau_f^2 E[b^2]$$

E. Penalty for no correction until t_f

$$\omega_{i0} = (\alpha_i \rho_i)(\alpha_i + \rho_i)^{-1}$$
$$\psi_{i0} = \psi_i + (m_i^*)^2$$
$$\lambda_{i0} = (c\tau_f)(\psi_{i0})^{-1/2}$$
$$g_{i0} = \exp - (q_1\lambda_{i0} + q_2\lambda_{i0}^2)$$
$$[p_i|0,f] = \omega_{i0} + \psi_{i0} g_{i0} + \psi_{i0} E[a^2] + \tau_f^2 E[b^2]$$

F. Test for no correction at t_i

If $[p_i|i,f] - [p_i|0,f] \geq 0$, make no correction. Go to time t_{i+1} and restart computations

If $[p_i|i,f] - [p_i|0,f] < 0$, continue

G. Predicted penalty for correction at t_{i+1}

$$\gamma_i = (\sigma_\theta v \tau_i)^2$$
$$\alpha_{i+1} = (\alpha_i \gamma_i)(\alpha_i + \gamma_i)^{-1}$$
$$\eta_{i+1}^{-1} = \eta_i^{-1} + \gamma_i^{-1}$$
$$\rho_{i+1} = (\eta_f \eta_{i+1})(\eta_{i+1} - \eta_f)^{-1}$$
$$\beta_{i+1} = \alpha_{i+1} + (m_i^*)^2 E[a^2] + (\tau_{i+1}^2) E[b]^2$$
$$\omega_{i+1} = (\beta_{i+1}\rho_{i+1})(\beta_{i+1} + \rho_{i+1})^{-1}$$
$$\psi_{i+1} = \beta_{i+1} - \omega_{i+1}$$
$$\lambda_{i+1} = [c - (m_i^*)(\tau_{i+1})^{-1}](\tau_f)(\psi_{i+1})^{-1/2}$$
$$g_{i+1} = \exp - (q_1\lambda_{i+1} + q_2\lambda_{i+1}^2)$$
$$[p_i|i+1,f] = \omega_{i+1} + \psi_{i+1} g_{i+1} + \psi_{i+1} E[a^2] + \tau_f^2 E[b^2]$$

H. Test for correction at t_i

If $[p_i|i,f] - [p_i|i+1,f] > 0$, make no correction. Go to time t_{i+1} and restart computations

If $[p_i|i,f] - [p_i|i+1,f] \leq 0$, continue (make correction)

I. Effect of correction at t_i

$$\Delta v_i = (m_i^*)(\tau_i)^{-1}$$
$$c = c - v_i$$
$$\alpha_i = \beta_i$$

go to time t_{i+1} and restart computations

plished: one at the final decision time t_f and another at some time $t_j < t_f$.

These constraints are suggested by present practice in the guidance of space vehicles, where reliability considerations demand that the guidance policy call for a minimum number of corrections. To reduce the probability of requiring added corrections, it seems best always to null all the error each time a correction is made. Since the resultant target estimate is then zero, it is reasonable to expect that no further corrections would be required before the final time; or, equivalently, that any such corrections and the corresponding proportional execution error and correction capability expenditure would be negligibly small.

With the total- and two-correction constraints, it follows from Eq. (23) that

$$\min p_i(\mathbf{x}_i) \leq \begin{cases} [p_i|0,f] \\ [p_i|i,f] \\ [p_i|i+1,f] \end{cases} \tag{28}$$

where the elements of inequality (28) are defined by Eqs. (2–4). The $[p_i|i,f]$ is calculated directly from Eq. (20). The $[p_i|i+1,f]$ is also calculated from Eq. (20) but with the state variables at t_{i+1} obtained from Eqs. (24–26); and the $[p_i|0,f]$, also calculated from Eq. (20), has the execution errors at t_i set equal to zero ($\beta_i = \alpha_i$) and

$$[\psi_i|0,f] = [\psi_i|i,f] + (m_i^*)^2 \tag{29}$$

Equation (29) calls upon the approximate equivalence between bias and standard deviation developed in Sec. IV. Thus, if the optimal guidance policy calls for a correction at time t_i, the following (necessary) condition must hold: if $d_i = 1$ is the optimal decision, then

$$\min p_i(\mathbf{x}_i) = [p_i|i,f] \leq \begin{cases} [p_i|i+1,f] \\ [p_i|0,f] \end{cases} \tag{30}$$

Equation (30) immediately establishes step 3 of the guidance policy described in Sec. II, where the decision $d_i = 0$ has arbitrarily been selected when the equality holds in order to minimize the number of corrections. Finally, it follows that, if $d_i = 1$ is the optimal decision, and $[p_i|i,f] < [p_i|0,f]$, then

$$s_i = [p_i|i,f] - [p_i|i+1,f] \leq 0 \tag{31}$$

Assuming there exists only one correction time $t_i < t_f$ which satisfies both Eqs. (30) and (31), the guidance policy described in Sec. II is established. For the explicit steps carried out in the determination of the decision at each time t_i, refer to Table 1.

It should be noted that this analysis determines the optimal decision at any time t_i, but not the value of the performance index which results from the decision. This value can only be determined from the complete dynamic programing analysis, or from a Monte Carlo simulation of the ensemble of all trajectories, where each decision d_i depends upon the random value of the state at t_i.

The two-correction constraint is readily established from reliability arguments, but the conditions that justify a total-correction policy need to be established. Thus, a total correction of the target error is approximately optimal if a two-correction policy is assumed and if 1) the time-to-go when the correction is made (τ_i) is large compared to the time-to-go at the final time (τ_f), and/or 2) the estimate at the decision time t_i is large compared to the standard deviation of estimate at the final time t_f.

This statement asserts that a total correction is approximately optimal if the correction is made relatively early and/or if the quality of the precorrection orbit determination data is significantly better than that of the postcorrection data. To show this, suppose that the effect of the proportional execution error is negligible and that a two-correction policy is assumed. If a partial correction is made at decision time t_i, the performance index becomes

$$[p_i|i,f] = \psi_i[\tfrac{1}{2}g(\lambda_i[+]) + \tfrac{1}{2}g(\lambda_i[-])] + \omega_i \tag{32}$$

where

$$\lambda_i(+) = \psi_i^{-1/2}\{[c_i - (d_i m_i^*/\tau_i)](\tau_f) + (1 - d_i)m_i^*\} \tag{33}$$

Table 2 Parameter values defining the idealized approach guidance problem

Symbol	Description	Value
T	Time from start to impact	10^6 sec
v	Spacecraft speed	5 km/sec
$(a_0)^{1/2}$	Standard deviation of a priori orbit determination error	10^3 km
Δt	Interval between decision times and tracking (star) observations	$5(10^3)$ sec
σ_θ	Standard deviation of noise on tracking (star) observations	10^{-3} rad
τ_f	Time from impact at final correction opportunity	55×10^3 sec
σ_a	Standard deviation of proportional execution errors	0.01
σ_b	Standard deviation of nonproportional execution errors	0.1 m/sec

$$\lambda_i(-) = \psi_i^{-1/2}\{[c_i - (d_i m_i^*/\tau_i)](\tau_f) - (1 - d_i)m_i^*\} \quad (34)$$

and, since the execution errors are to be neglected, ψ_i is given by Eq. (12). If the correction is to be optimal, it is necessary that $(\partial p_i/\partial d_i) = 0$, which in the interval $0 \leq d_i \leq 1$ can be expanded in Taylor series about $d_i = 1$ to obtain

$$0 = [(\partial p_i/\partial d_i)|d_i = 1] + [(\partial^2 p_i/\partial d_i^2)|d_i = 1]\Delta d_i + \ldots \quad (35)$$

where $\Delta d_i = (d_i - 1)$. But, from Eqs. (32–34),

$$\left[\left(\frac{\partial p_i}{\partial d_i}\right)\bigg| d_i = 1\right] = \left[(4m_i^*)(\psi_i)^{1/2}\left(\frac{\tau_f}{\tau_i}\right)\right] \times \left[\int_\lambda^\infty f(z)(z - \lambda)dz\right] \quad (36)$$

$$\left[\left(\frac{\partial^2 p_i}{\partial d_i^2}\right)\bigg| d_i = 1\right] = [4m_i^{*2}]\left[\left(\frac{\tau_f}{\tau_i}\right)^2 + 1\right]\left[\int_\lambda^\infty f(z)dz\right] \quad (37)$$

Solving for Δd_i from Eq. (35),

$$\Delta d_i = -\left(\frac{\psi_i}{m_i^{*2}}\right)^{1/2}\left(\frac{\tau_f}{\tau_i}\right)\left[\int_\lambda^\infty f(z)\;(z - \lambda)dz\right] \times \left\{\left[1 + \left(\frac{\tau_f}{\tau_i}\right)^2\right]\int_\lambda^\infty f(z)dz\right\}^{-1} \quad (38)$$

Equation (38) shows that $d_i \to 1$ as $\psi_i/m_i^{*2} \to 0$ and/or $\tau_f/\tau_i \to 0$, which establishes the stated result. [From l'Hospital's rule, it can be verified that the ratio of the integral terms in Eq. (38) goes to zero as λ goes to infinity.]

An alternate approach to justifying the total-correction guidance policy is to treat the residual target error to be corrected at the final time as the sum of the absolute values of the error left uncorrected at t_i (because $d_i < 1$) and the random error accumulated between t_i and t_f. This treatment leads to the conclusion that an optimal two-correction policy is always a total-correction policy, but would yield a pessimistic value for the correction capability utilized by a partial-correction policy.

VII. Depletion Mode of Operation

It is assumed in the foregoing that at each decision time t_i there is sufficient propulsion capability to perform a total correction, and that an unlimited number of corrections can be made during the remainder of the mission. Neither of these conditions is guaranteed, however, for it is possible to deplete the propellant reserves, and engineering constraints may limit the total number of corrections.

Definition 3: The depletion mode of operation occurs at t_i when

$$n < 2 \text{ and/or } c < m_i^*/\tau_i$$

where n is the total number of corrections that can be performed at decision times $t_i, t_{i+1}, \ldots, t_f$.

Without further justification, the following intuitively obvious depletion policy will be adopted: the optimal policy for the depletion mode of operation is to correct as much of the error as possible at t_i when $s_i \leq 0$, where

$$s_i = [\beta_i + (r_i)^2] - [\beta_{i+1} + (r_{i+1})^2]$$

and

$$r_i = \begin{cases} 0 & \text{if } c\tau_i \geq m^* \\ m^* - c\tau_i & \text{if } c\tau_i < m^* \end{cases}$$

where c = correction capability at t_i, m^* = estimate of target error at t_i, and β_i = uncertainty resulting from orbit determination and execution errors, assuming a correction only at t_i. The quantities r_{i+1} and β_{i+1} are similarly defined. Notice that n effectively becomes a new state variable.

VIII. Extension to Multiple Dimensions

The analysis has, thus far, considered only the simple case in which one miss component need be dealt with; but, in general, it is necessary to estimate all random variables that affect the observed data in order to obtain a minimum-variance estimate of the orbit parameters. Thus, all position and velocity components must be estimated, as well as unknown biases in the measuring devices and errors in the physical constants that describe the mathematical model. It is also necessary to consider more than one miss component in order to compute the probability of impacting the target area. This general case can be treated in the manner presented previously, however, by interpreting the variances associated with the idealized problem as being traces of certain combinations of covariance matrices. In this way, a corresponding one-dimensional problem is constructed. The justification for this approach will not be rigorously established, but it is intuitively clear that the guidance policy so constructed is reasonable.

If Γ_i is the covariance matrix describing the error in the total estimate vector at t_i, and if there are not corrections in the interval t_i, t_{i+k}, the covariance of the error in the total estimate vector at t_{i+k} is

$$\Gamma_{i+k} = \left[\Gamma_i^{-1} + \sum_{j=1}^{k-1} J_j\right]^{-1} \quad (39)$$

where J_j is the generalized inverse (normal matrix) of the covariance matrix describing the error in estimate due to observations gathered in the interval (t_j, t_{j+1}). If a correction is accomplished at t_i, the covariance matrix Γ_i is replaced with

$$\Lambda_i = \Gamma_i + E[\delta\mathbf{v}_i\delta\mathbf{v}_i^T] \quad (40)$$

where $E[\delta\mathbf{v}_i\delta\mathbf{v}_i^T]$ is the covariance added by the random velocity execution errors (superscript T indicates transpose).

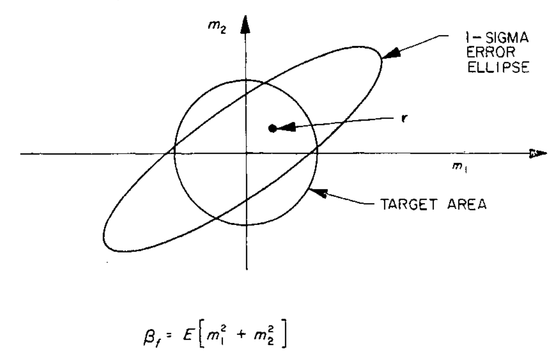

Fig. 5 Two-dimensional target error.

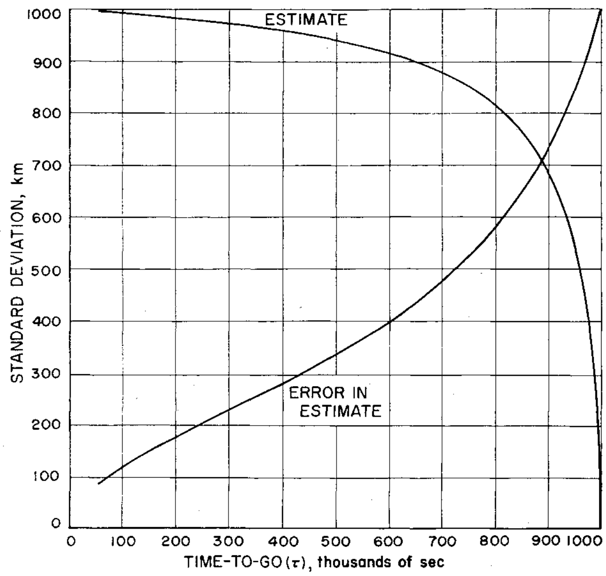

Fig. 6 Standard deviation of estimate and error in estimate vs time-to-go, assuming no corrections.

Let **m** be the n dimensional target error vector that is to be nulled, and define the following relationships:

$$m_i^* = |\mathbf{m}_i^*| \tag{41}$$

$$\Delta v_i = |[\partial \mathbf{m}/\partial \mathbf{v}_i]^{-1}\mathbf{m}_i^*| \tag{42}$$

$$\alpha_i = \begin{matrix}\text{trace of } \mathbf{m}\\ \text{component}\\ \text{submatrix of}\end{matrix}\ [\Gamma_i] \tag{43}$$

$$\beta_i = \begin{matrix}\text{trace of } \mathbf{m}\\ \text{component}\\ \text{submatrix of}\end{matrix}\ [\Lambda_i] \tag{44}$$

$$\omega_i = \begin{matrix}\text{trace of } \mathbf{m}\\ \text{component}\\ \text{submatrix of}\end{matrix}\ \left[\Lambda_i^{-1} + \sum_{j=i}^{i_f-1} J_i\right]^{-1} \tag{45}$$

$$\psi_i = \beta_i - \omega_i \tag{46}$$

The quantity $E[r^2]_i$ can be determined for the general case by evaluating a multiple integral. If the variances of the individual components of the estimate of the target error at t_f are all equal, for example, it follows that

$$E[r^2]_i = \left(\frac{2}{\pi}\right)^k \left(\frac{\psi_i}{n}\right) \int_{\lambda_i}^{\infty} (z - \lambda_i)^2 (z^{n-1}) \exp\left(\frac{-z^2}{2}\right) dz$$

where $n = 1$, 2, or 3 is the dimension of **m**, and $k = |(n - 2)/2|$. With these relationships established, the analysis proceeds as in the one-dimensional case. The two-dimensional case is pictured in Fig. 5.

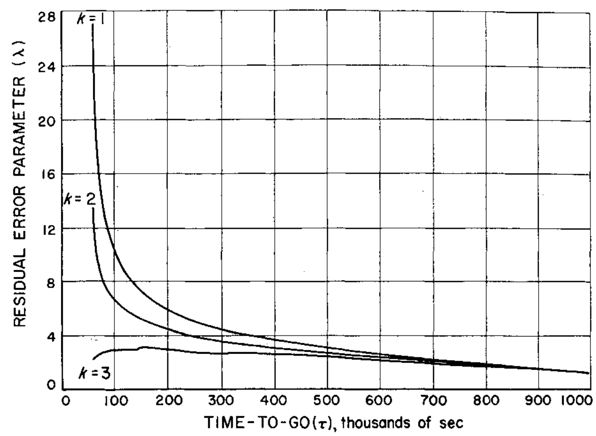

Fig. 8 Residual error parameter vs time-to-go.

IX. Application of the Guidance Policy

The guidance policy developed previously was applied to a numerical example in order to demonstrate its effectiveness. The mathematical model describing the system was as given in Sec. III, with the parameters defining the problem chosen so as to represent reasonably a typical Mars-approach guidance situation (Table 2). For example, the final time t_f of approximately 15 hr before impact might correspond to the splitting off of an entry capsule from the spacecraft. To avoid a Monte Carlo simulation, a k-sigma case was constructed by assuming that the estimated target error at each time t_i was k times the standard deviation of the estimate (over the ensemble of all experiments). The switching function was computed by using this simulated value. Thus, initially the estimate would be zero (at $t_0 = 0$); as tracking data were gathered, it would build asymptotically toward $k(\alpha_0)^{1/2}$, be corrected to zero at the first correction time, and the process would then be reinitiated with β_i replacing α_0. It was assumed that the correction capability initially was 20 m/sec, this number being chosen to adequately handle the 3-sigma case. The computer program developed to do this analysis is described in the Nomenclature and Table 1. For the computations, $g(\lambda)$ was approximated by

$$g(\lambda) = \exp-(q_1\lambda + q_2\lambda^2) \tag{47}$$

Table 3 Summary of results for four representative cases[a]

Case	Sigma level, k	Correction number	Time-to-go at correction, sec × 10^{-3}	Correction applied, m/sec	Total correction applied, m/sec	Final rms error ($p^{1/2}$), km
1	0.1	1	55	1.82	1.82	87.20
2	1	1	390	2.47		
		2	55	4.76	7.23	87.32
3	2	1	355	5.79		
		2	55	8.43	14.22	87.45
4	3	1	315	9.26		
		2	150	3.71		
		3	55	6.66	19.63	87.63

[a] Total correction capability used constrained to be less than 20 m/sec.

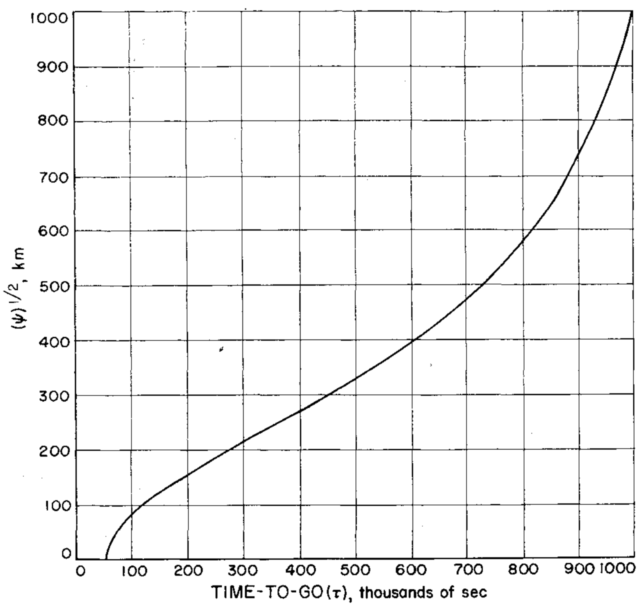

Fig. 7 Standard deviation of estimate at t_f, assuming a correction at τ, for 1-, 2-, and 3-sigma levels.

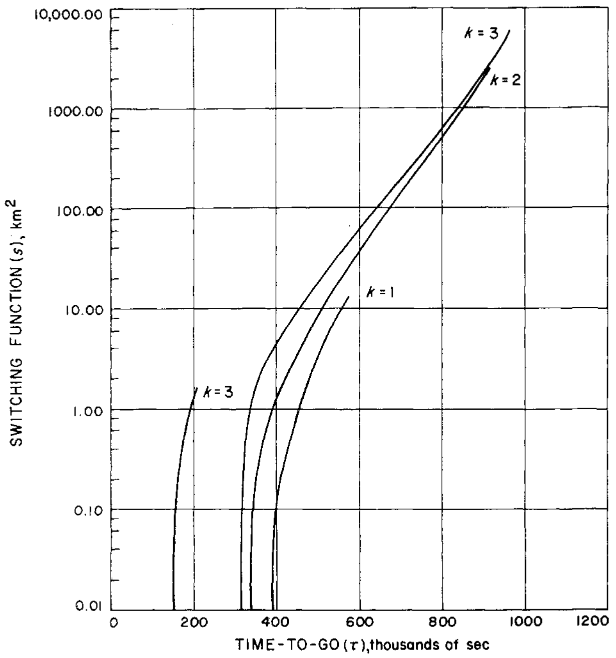

Fig. 9 Switching function vs time-to-go for various sigma levels.

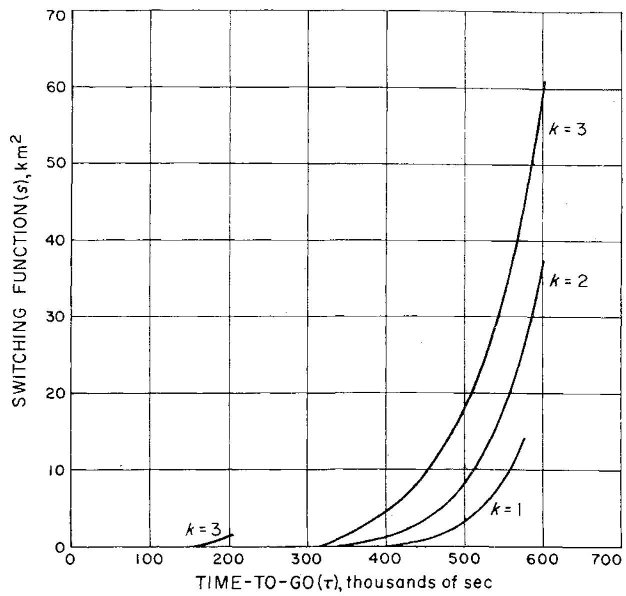

Fig. 10 Magnified view of switching function for various sigma levels.

where $q_1 = 1.5641$ and $q_2 = 0.36336$. The orbit determination statistics, assuming no corrections, are described in Fig. 6. The results for the 0.1-, 1-, 2-, and 3-sigma cases are presented in Table 3 and Figs. 7–10.

X. Conclusion

An adaptive guidance correction policy has been developed which minimizes the expected value of the squared target error, subject to the constraint that the total propellant expenditure be less than some specified amount. This is a good criterion for missions terminating at the final time, for then the best accuracy must be obtained, and there is no particular advantage in finishing with propellant left over. The scheme is simple enough for use in the real-time operational situation. Although the analysis has been carried out only for the idealized case, an extension to the general case has been suggested.

Computational difficulties inherent in the dynamic programing approach to the problem have been avoided by showing that the guidance policy can be developed in terms of the instantaneous state of the system. This simplification was a consequence of the constraints that either a total correction or no correction is to be made at each decision time, and that a two-correction policy is to be employed. Such restrictions are imposed upon present-day guidance logic for unmanned lunar and interplanetary spacecraft, because each correction degrades the reliability of the spacecraft, disturbing it from the normal cruise mode and subjecting it to potential failures in the command and execution subsystem. The simulation results presented here verify that a minimum number of corrections are called for; only in the 3-sigma case are more than two corrections made. Theoretical discussions of some properties of the unconstrained case are given in Refs. 7 and 8.

The analysis presented here has been extended by D. W. Curkendall of the Jet Propulsion Laboratory to treat the actual (not approximate) probability of mission success as the performance index, and to include the effect of manuever reliability. A complete Monte Carlo simulation was performed to evaluate the guidance policy.[9] The total- and two-correction constraints were still applied, however, and further study is needed in order to determine the precise effect of these constraints. The complete dynamic programing analysis discussed in Sec. V can be carried out for the idealized guidance problem without excessive use of computing machine time, and some numerical results have been obtained using this approach. Indeed, it was this effort which suggested the simplified analysis presented here. More work with the unconstrained dynamic programing formulation is planned.

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